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Canonical quantization, spacetime noncommutativity and deformed symmetries in field theory

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Abstract
Within the spirit of Dirac's canonical quantization, noncommutative spacetime field theories are introduced by making use of the reparametrization invariance of the action and of an arbitrary non-canonical symplectic structure. This construction implies that the constraints need to be deformed, resulting in an automatic Drinfeld twisting of the generators of the symmetries associated with the reparametrized theory. We illustrate our procedure for the case of a scalar field in (1+1)-spacetime dimensions, but it can be readily generalized to arbitrary dimensions and arbitrary types of fields.

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1. Introduction

Even though deformation quantization has been developed for more than 30 years, mostly in the mathematical literature (see, e.g., the nice review of the history, developments and bifurcations of the field by Sternheimer in [1]), a large degree of the more recent involvement of physicists in the field is due to the seminal work by Seiberg and Witten [2] who showed how noncommutative field theories arise as low energy limits of open string theory. For a review of the physical literature see, e.g., [3].

The term spacetime noncommutativity is widely used in field theory as a convenient way to describe a special type of interaction consisting in mathematically deforming the product in the algebra of field functions. However, since the arguments of the fields are the parameters of the theory, there is very little physical basis for speaking about noncommutativity of these parameters.

One can however find more physical substance to that designation if one recalls that quantum mechanics can be viewed as a minisuperspace of field theory, where most of the degrees of freedom have been frozen, and that observables there are represented by Hermitian operators acting on a Hilbert space, so that noncommutativity of the dynamical variables of the system is readily understood then as the noncommutativity of their corresponding operators. In this way, the physical argument that measurements below distances of the order of the Planck length lose operational significance can be mathematically described by extending the usual Heisenberg algebra of ordinary quantum mechanics to one including the noncommutativity of the operators related to the spacetime dynamical variables. Moreover, as we have shown elsewhere [4], the $\star$-product deformation of functions of spacetime then naturally results from the Weyl–Wigner–Gröenewold–Moyal (WWGM) formalism of quantum mechanics when considering in this extended context the algebra of the Weyl-equivalent functions corresponding to operator functions of the Heisenberg space and time operators.

Other approaches for constructing a noncommutative spacetime quantum mechanics have been based on the idea of promoting the time parameter to the rank of a coordinate by means of a reparametrization, whereby time becomes a function $t(\tau)$ of a new parameter τ and thus becomes a coordinate on the same level as the spatial coordinates $x^i(\tau)$, either by fixing the gauge degrees of freedom [5–7] or by deforming the symplectic structure of the theory [8]. An important feature of these formulations is that, because additional degrees of freedom are added to the original theory, first-class constraints appear in the reparametrized theory. In order to eliminate these additional degrees of freedom one can apply gauge conditions or follow Dirac's quantization method and operate with the constraints on the state vectors in order to obtain the physical states of the system.

Now, when going on to field theory both the time and space coordinates play the role of parameters of the field, so applying commutation relations to them is, to say the least, even more unclear; as it is the relation of this procedure to the operator spacetime noncommutativity in quantum mechanics, particularly when we view the latter as a minisuperspace of the former and in the light of what we have just said above.

In order to shed some additional insight into some of these issues, we explore in the present work how the above-referred reparametrization formalism can be extended to the case of field theory on a noncommutative spacetime. However, since we are now dealing with a system with an infinite number of degrees of freedom, the basic idea here is to promote the coordinates of the spacetime, that are the parameters on which the field depends, to new fields in the ensuing reparametrized theory. This idea is not new in the case of commutative spacetime. For example in [9] such a construction of a field theory was used as a model when considering the canonical quantization of gravity. Making use of the results in that work, it is possible to construct the reparametrized theory for any field theory, with as many constraints as the number of coordinate fields being added.

Moreover, as it occurs in the case of general relativity, the parametrized field theory is also invariant under diffeomorphisms, so such a construction provides an ideal arena for studying these symmetries at the quantum level there. It is interesting to note that this idea was also used in the context of string theory as a means for constructing a theory which would be independent of the background [10], and it also appears in the context of non-relativistic strings [11].

Once the spacetime coordinates are promoted to the rank of fields, it does make sense to impose commutation relations among them. This can be achieved by deforming the symplectic structure in the original theory and thus arriving at a noncommutative field theory. Such a theory is already at the first quantization level radically different from the usual one, because—since the coordinate fields do not commute—we cannot use their eigenstates as configuration space bases to construct amplitudes of the state vector, which will then necessarily have to be either functions of both the eigenvalues of the momenta field operators as well as of some of the coordinate fields (those that commute among themselves) or only of the eigenvalues of the commuting momenta fields.

Another important point that we analyze in this paper is the deformed symmetries that appear in the noncommutative theory. According to our procedure, the nature of these deformed symmetries appears automatically since, when deforming the symplectic structure the algebra of the constraints is broken and, in order to preserve it, it is necessary to deform the generators of the symmetry by means of what turns out to be a Drinfeld twist. The algorithm suggested by our procedure for this twist is quite straightforward to implement and can be readily generalized to other types of $\star$-products, as well as to situations where noncommutativity involves both spacetime and momenta variables. It also provides a framework for investigating the relations of the isometries of more general curved noncommutative backgrounds to possible different $\star$-product deformations of the multiplication in the space of functions on which the spacetime diffeomorphisms act.

We should stress here that our formalism can be implemented without difficulty when dealing with spacial noncommutativity only. However in the explicit example discussed in section 2 we assumed for computational simplicity a two-dimensional spacetime manifold, so obviously one of the noncommuting coordinates had to be the time. The generalization of the formalism to higher dimensional spaces is given at the end of section 3. One should be aware though that in theories where the time coordinate is also noncommutative there are problems related to unitarity, see, e.g., [12, 13]. Nonetheless, in analogy to other cases discussed in the literature [14–16], it might be possible to overcome this obstacle by proposing an appropriate time ordering or a scalar product that makes the theory unitary.

2. Spacetime noncommutativity in field theory

In a previous paper [8], noncommutative spacetime quantum-mechanical theories were constructed by using a reparametrization invariant action where the time parameter is elevated to the rank of a dynamical variable. Furthermore, in order to consider the noncommutativity between the spacetime coordinates, an arbitrary non-canonical symplectic structure was introduced that, together with Dirac's Hamiltonian method, led to Dirac brackets for the spacetime dynamical variables, which when quantized can be interpreted as noncommutative. As mentioned in the introduction, we shall apply this procedure to the case of fields in order to investigate the implications of noncommutativity of spacetime as field variables on the algebra of the reparametrized fields.

2.1. Reparametrization of the scalar field

To illustrate the procedure, consider for simplicity the case of a scalar field in a $(D+1)$-dimensional Minkowski spacetime $\mathcal{M}$ with signature $(1, -1, \ldots, -1)$ and with a potential $V(\phi)$. The corresponding action is then

$$S = \int \mathrm{d}x\, \mathrm{d}t \left(\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right). \tag{2.1}$$

In order to parameterize the full spacetime, let us write

$$t = t(\tau, \boldsymbol{\sigma}), \qquad x^i = x^i(\tau, \boldsymbol{\sigma}), \tag{2.2}$$

so that the new action in terms of the new parameters $\tau, \boldsymbol{\sigma}$ reads

$$S = \int \mathrm{d}\tau\, \mathrm{d}^D\sigma \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right), \tag{2.3}$$

with the inverse metric $g^{\alpha\beta}$ given by

$$g^{\alpha\beta} = \frac{\partial\sigma^\alpha}{\partial x^\mu}\frac{\partial\sigma^\beta}{\partial x_\mu} \tag{2.4}$$

and $g := \det(g_{\mu\nu})$ where $\sqrt{-g} = J$ is the Jacobian of the transformation $g := \det(g_{\mu\nu})$. Also, in (2.3) we are making the identification $\partial_0 \equiv \partial_\tau$ and $\partial_i = \partial_{\sigma_i}$.

The canonical momentum associated with the field ϕ is

$$P_\phi = J\frac{\partial\tau}{\partial x^\mu}\frac{\partial\sigma^\alpha}{\partial x_\mu}\frac{\partial\phi}{\partial\sigma^\alpha}, \qquad \sigma^0 = \tau, \qquad \sigma^i \equiv \sigma^i, \tag{2.5}$$

and, following [9], we define the canonical momenta associated with the spacetime coordinates as

$$p_\nu \equiv -J\frac{\partial\tau}{\partial x^\mu}T^\mu{}_\nu, \tag{2.6}$$

where $T^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\big(\frac{1}{2}\partial^\rho\phi\partial_\rho\phi - V(\phi)\big)$ is the unparametrized energy–momentum tensor of the field. In terms of this momenta the Hamiltonian action becomes

$$S = \int \mathrm{d}\tau\,\mathrm{d}^D\sigma\left(P_\phi\dot\phi + p_\mu\dot x^\mu - \lambda^\nu\left(p_\nu + J\frac{\partial\tau}{\partial x^\mu}T^\mu{}_\nu\right)\right), \tag{2.7}$$

where we have introduced the definition of the momenta (2.6) as Hamiltonian constraints due to the fact that the right hand side of (2.6) is independent of the velocities when the energy–momentum tensor is expressed as a function of the canonical variables ϕ, P_ϕ [9].

We can write an alternate expression for action (2.7)—based on the foliation $\Sigma\times\mathbb{R}$ of spacetime, where $\mathbb{R}$ is the temporal direction and Σ is a space-like hypersurface of constant τ—by introducing the vectors $\mathbf{s}_i$ tangent to Σ, with components $s^\mu_i = \partial_{\sigma^i}x^\mu$, and their duals $\hat{\mathbf{s}}^i$, with components s^i_α such that $s^i_\alpha s^\alpha_j = \delta^i_j$, and also introducing the unit vector $\hat{\mathbf{n}}$, normal to this hypersurface, with components

$$n^\mu = \left(\sqrt{g^{00}}\dot x^\mu + \frac{g^{0i}}{\sqrt{g^{00}}}\frac{\partial x^\mu}{\partial\sigma^i}\right), \qquad i = 1,\ldots,d, \tag{2.8}$$

such that $n^\mu n_\mu = 1$.

We can then write the (D+1)-vector constraint $\boldsymbol{\Pi}$, with components $\Pi_\nu \equiv p_\nu + J\frac{\partial\tau}{\partial x^\mu}T^\mu{}_\nu$, as

$$\boldsymbol{\Pi} \equiv (\hat{\mathbf{n}}\hat{\mathbf{n}} + \hat{\mathbf{s}}^i\mathbf{s}_i)\cdot\boldsymbol{\Pi} = \hat{\mathbf{n}}\mathcal{H}_\perp + \hat{\mathbf{s}}^i\mathcal{H}_j, \tag{2.9}$$

where $(\hat{\mathbf{n}}\hat{\mathbf{n}} + \hat{\mathbf{s}}^i\mathbf{s}_i)$ is the unit dyadic, multiplication is with the Lorentzian metric,

$$\begin{aligned}\mathcal{H}_\perp &:= \hat{\mathbf{n}}\cdot\boldsymbol{\Pi} = n^\mu\left(p_\mu + J\frac{\partial\tau}{\partial x^\nu}T^\nu{}_\mu\right)\\ &= \frac{1}{2\sqrt{-\gamma}}\left(P^2_\phi + \gamma\gamma^{ij}\partial_{\sigma^i}\phi\partial_{\sigma^j}\phi\right) + n^\mu p_\mu + \sqrt{-\gamma}V(\phi),\end{aligned} \tag{2.10}$$

$$\mathcal{H}_j := \mathbf{s}_j\cdot\boldsymbol{\Pi} = (\partial_{\sigma^j}x^\mu)\left(p_\mu + J\frac{\partial\tau}{\partial x^\nu}T^\nu{}_\mu\right) = P_\phi\partial_{\sigma^j}\phi + p_\mu\partial_{\sigma^j}x^\mu, \tag{2.11}$$

and where $\gamma_{ij} \equiv g_{ij}$ is the D-metric of the Σ-hypersurface, γ^{ij} is the inverse matrix to γ_{ij} and γ is the determinant of γ_{ij}. Inserting now (2.9) into (2.7) we can write

$$S = \int \mathrm{d}\tau\,\mathrm{d}^D\sigma(P_\phi\dot\phi + p_\mu\dot x^\mu - N\mathcal{H}_\perp - N^i\mathcal{H}_i), \tag{2.12}$$

after identifying the projections $\boldsymbol{\lambda} \cdot \hat{\mathbf{n}}$, $\boldsymbol{\lambda} \cdot \hat{\mathbf{s}}^i$ of the Lagrange multipliers with the lapse and shift functions N and N^i, respectively. $\mathcal{H}_\perp$ is the super-Hamiltonian and $\mathcal{H}_i$ are the super-momenta for the system.

The Poisson brackets of these super-Hamiltonian and super-momenta are given by [17]

$$\begin{aligned}
\{\mathcal{H}_\perp(\boldsymbol{\sigma},\tau),\mathcal{H}_\perp(\boldsymbol{\sigma}',\tau\} &= \sum_{i=1}^{D}(\mathcal{H}^i(\boldsymbol{\sigma},\tau)+\mathcal{H}^i(\boldsymbol{\sigma}',\tau))\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),\\
\{\mathcal{H}_i(\boldsymbol{\sigma},\tau),\mathcal{H}_k(\boldsymbol{\sigma}',\tau)\} &= (\mathcal{H}_k(\boldsymbol{\sigma},\tau)\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}')+\mathcal{H}_i(\boldsymbol{\sigma}',\tau))\partial_{\sigma^k}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),\\
\{\mathcal{H}_i(\boldsymbol{\sigma},\tau),\mathcal{H}_\perp(\boldsymbol{\sigma}',\tau)\} &= (\mathcal{H}_\perp(\boldsymbol{\sigma},\tau))\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),
\end{aligned} \tag{2.13}$$

from where we see that the constraints are first class.

Let us now further simplify the calculations and the basic steps leading to a noncommutative field theory by first considering our scalar field to be propagating in a flat spacetime with Minkowskian coordinates (t, x) and signature $(1, -1)$. In this case,

$$g^{\mu\nu} = g^{-1}\begin{pmatrix} t'^2 - x'^2 & -(t'\dot{t} - x'\dot{x}) \\ -(t'\dot{t} - x'\dot{x}) & \dot{t}^2 - \dot{x}^2 \end{pmatrix} \tag{2.14}$$

and

$$g := \det(g_{\mu\nu}) = -(\dot{t}x' - \dot{x}t')^2, \tag{2.15}$$

where the primes denote partials with respect to σ while the dots are partials with respect to τ.

Explicit expressions for the momenta canonical to t, x and ϕ can be derived from (2.5) and (2.6) or, even simpler, directly from (2.3), (2.14) and (2.15). They are given by

$$\begin{aligned}
p_t &= -\frac{1}{\sqrt{-g}}(\dot{t}\phi'^2 - t'\phi'\dot{\phi}) - x'V(\phi) - \frac{x'}{2g}[(t'^2 - x'^2)\dot{\phi}^2 - 2(t'\dot{t} - x'\dot{x})\phi'\dot{\phi} + (\dot{t}^2 - \dot{x}^2)\phi'^2],\\
p_x &= \frac{1}{\sqrt{-g}}(\dot{x}\phi'^2 - x'\phi'\dot{\phi}) + t'V(\phi) + \frac{t'}{2g}[(t'^2 - x'^2)\dot{\phi}^2 - 2(t'\dot{t} - x'\dot{x})\phi'\dot{\phi} + (\dot{t}^2 - \dot{x}^2)\phi'^2],\\
P_\phi &= -\frac{1}{\sqrt{-g}}[(t'^2 - x'^2)\dot{\phi} - (t'\dot{t} - x'\dot{x})\phi'].
\end{aligned} \tag{2.16}$$

From these expressions it can be readily verified that

$$p_t\dot{t} + p_x\dot{x} + P_\phi\dot{\phi} = \mathcal{L} = \sqrt{-g}\big(\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\big). \tag{2.17}$$

Furthermore, because we are introducing the fields $t(\tau, \sigma)$ and $x(\tau, \sigma)$ as new degrees of freedom, the theory must have constraints in the Hamiltonian formalism. Specifically, since instead of our two original phase-space degrees of freedom we now have six, we thus need four relations which we can get by two primary first-class constraints and two gauge conditions.

The primary constraints follow from specializing (2.10) and (2.11) to the case $D = 1$ and are explicitly given by

$$\begin{aligned}
\mathcal{H}_\perp &= (-\gamma)^{-\frac{1}{2}}\big[\tfrac{1}{2}\big(P_\phi^2 + \phi'^2\big) + p_t x' + p_x t' + (x'^2 - t'^2)V(\phi)\big] \approx 0,\\
\mathcal{H}_1 &= p_x x' + p_t t' + P_\phi\phi' \approx 0.
\end{aligned} \tag{2.18}$$

Defining

$$\mathcal{H}_{(\perp,1)}[f] := \int \mathrm{d}\sigma f(\sigma)\mathcal{H}_{(\perp,1)}(\sigma,\tau), \tag{2.19}$$

it can then be shown that

$$\begin{aligned}
\{\mathcal{H}_\perp[f],\mathcal{H}_\perp[g]\} &= \mathcal{H}_1[fg' - gf'],\\
\{\mathcal{H}_1[f],\mathcal{H}_1[g]\} &= \mathcal{H}_1[fg' - gf'],\\
\{\mathcal{H}_\perp[f],\mathcal{H}_1[g]\} &= \mathcal{H}_\perp[fg' - gf'].
\end{aligned} \tag{2.20}$$

Moreover, since the test functions f and g are arbitrary, we can take the functional derivatives of (2.20) relative to them to arrive at

$$\begin{aligned}
\{\mathcal{H}_\perp(\sigma,\tau),\mathcal{H}_\perp(\sigma',\tau\} &= (\mathcal{H}_1(\sigma,\tau)+\mathcal{H}_1(\sigma',\tau))\delta'(\sigma-\sigma'),\\
\{\mathcal{H}_1(\sigma,\tau),\mathcal{H}_1(\sigma',\tau)\} &= (\mathcal{H}_1(\sigma,\tau)+\mathcal{H}_1(\sigma',\tau))\delta'(\sigma-\sigma'),\\
\{\mathcal{H}_1(\sigma,\tau),\mathcal{H}_\perp(\sigma',\tau)\} &= (\mathcal{H}_\perp(\sigma,\tau))\delta'(\sigma-\sigma'),
\end{aligned} \tag{2.21}$$

where $\delta'(\sigma-\sigma') := \partial_\sigma\delta(\sigma-\sigma')$, which reproduce (2.13) for the case $D=1$. Note that these constraints close in the constant τ Poisson brackets according to the Virasoro algebra without a central charge and they are first class, as we already know. But first-class constraints are generically associated with gauge invariance, which in this case is the invariance of action (2.3) under two-dimensional reparametrizations, with its generators satisfying algebra (2.21).

Also, since $H=\int \mathrm{d}\sigma\,(N\mathcal{H}_\perp + N^1\mathcal{H}_1)$ is the Hamiltonian of the theory, it clearly follows that

$$\dot{\mathcal{H}}_{(\perp,1)} = \{\mathcal{H}_{(\perp,1)}, H\} \approx 0, \tag{2.22}$$

so the constraints are preserved by the 'time' τ evolution.

Next, in order to introduce spacetime noncommutativity in the Dirac quantization procedure for the above theory, we need to implement an additional general symplectic structure into our formalism.

2.2. Symplectic structure

For this purpose consider the following general first-order action:

$$S = \int \mathrm{d}\tau\,\mathrm{d}\sigma\,(A_a(z)\dot{z}^a - N\tilde{\mathcal{H}}_\perp - N^1\tilde{\mathcal{H}}_1), \tag{2.23}$$

with symplectic variables $z^a = (t, x, \phi, p_t, p_x, P_\phi)$. Here, $\tilde{\mathcal{H}}_\perp$ and $\tilde{\mathcal{H}}_1$ are weakly zero and appropriately modified first-class constraints to be specified below. The six potentials A_a play the role of momenta canonically conjugate to z^a. Action (2.23) allows us to generate an arbitrary symplectic structure associated with the Poisson brackets in the Hamiltonian formulation, but in order that it be equivalent to action (2.12) for $D=1$, we need six additional second-class primary constraints (these, together with the two first-class constraints and their corresponding two compatibility conditions, give the relations needed to eliminate ten of the twelve degrees of freedom in z^a's).

The additional second-class constraints follow by noting that the canonical momenta conjugate to z^a are given by

$$\pi_{z_a} = \partial_{\dot{z}^a}(A_a(z)\dot{z}^a - N\tilde{\mathcal{H}}_\perp - N^1\tilde{\mathcal{H}}_1) = A_a(z), \tag{2.24}$$

and since they are independent of the velocities they lead to the constraints

$$\chi_a = \pi_{z_a} - A_a \approx 0. \tag{2.25}$$

Hence, the action of our constrained system is now given by

$$S = \int \mathrm{d}\tau\,\mathrm{d}\sigma\,(A_a(z)\dot{z}^a - \mathcal{H}_T), \tag{2.26}$$

with

$$\mathcal{H}_T = N\tilde{\mathcal{H}}_\perp + N^1\tilde{\mathcal{H}}_1 + \mu^a\chi_a. \tag{2.27}$$

Note that from (2.25) we have

$$\{\chi_a, \chi_b\} = \frac{\partial A_b}{\partial z_a} - \frac{\partial A_a}{\partial z_b} := \omega_{ab}, \tag{2.28}$$

so the constraints χ_a are indeed second class (the Poisson brackets here are to be evaluated in the extended phase space (z^a, π_a)).

Moreover, in order that the consistency conditions

$$\dot{\chi}_a = \left\{\chi_a, \int \mathrm{d}\sigma \mathcal{H}_T\right\} = -N\frac{\partial\tilde{\mathcal{H}}_\perp}{\partial z^a} - N^1\frac{\partial\tilde{\mathcal{H}}_1}{\partial z^a} + \mu^b\omega_{ab} \approx 0, \tag{2.29}$$

$$\dot{\tilde{\mathcal{H}}}_{\perp,1} = \left\{\tilde{\mathcal{H}}_{\perp,1}, \int \mathrm{d}\sigma \mathcal{H}_T\right\} = \mu^a\left\{\tilde{\mathcal{H}}_{\perp,1}, \int \mathrm{d}\sigma\, \chi_a\right\} \approx 0 \tag{2.30}$$

be satisfied, we need, solving (2.29) for μ^a, that

$$\mu^a = \omega^{ab}\left(N\frac{\partial\tilde{\mathcal{H}}_\perp}{\partial z^b} + N^1\frac{\partial\tilde{\mathcal{H}}_1}{\partial z^b}\right), \tag{2.31}$$

and also that

$$\omega^{ab}\left(\frac{\partial\tilde{\mathcal{H}}_\perp}{\partial z^a}\frac{\partial\tilde{\mathcal{H}}_1}{\partial z^b}\right) \approx 0, \tag{2.32}$$

which results from inserting (2.31) into (2.30) and using the arbitrariness of the Lagrange multipliers.

Introducing now the Dirac brackets

$$\{\xi, \rho\}^* := \{\xi, \rho\} - \{\xi, \chi_a\}\omega^{ab}\{\chi_b, \rho\}, \tag{2.33}$$

it readily follows that

$$\{\tilde{\mathcal{H}}_\perp, \tilde{\mathcal{H}}_1\}^* = \omega^{ab}\left(\frac{\partial\tilde{\mathcal{H}}_\perp}{\partial z^a}\frac{\partial\tilde{\mathcal{H}}_1}{\partial z^b}\right). \tag{2.34}$$

Hence, in order to satisfy the compatibility condition (2.30) we need to choose our modified constraints $\tilde{\mathcal{H}}_\perp, \tilde{\mathcal{H}}_1$ such that their Dirac bracket is weakly zero. We shall defer the proof that such a choice indeed exist for later on, and note at this point that

$$\{\chi_a, \chi_b\}^* = 0. \tag{2.35}$$

We can therefore treat χ_a as strongly zero in our formalism, after replacing the Poisson brackets by the Dirac brackets. Note also that (2.33) implies

$$\{z^a, z^b\}^* = \omega^{ab}, \tag{2.36}$$

and by assuming further that the symplectic structure is determined by

$$\omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & \theta & 0 \\ 0 & 1 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \omega^{ab} = \begin{pmatrix} 0 & \theta & 0 & 1 & 0 & 0 \\ -\theta & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.37}$$

we find that (2.37) incorporates spacetime noncommutativity into the formalism. In particular upon quantization the strong equations $\chi_a = 0$ need to be promoted to a relation between quantum operators:

$$\hat{\pi}_{z_a} - \hat{A}_a = 0, \tag{2.38}$$

and we have from (2.36) that at equal τ

$$\begin{aligned}
&[\hat{t}(\tau,\sigma),\hat{x}(\tau,\tilde{\sigma})] = \mathrm{i}\theta\delta(\sigma-\tilde{\sigma}),\\
&[\hat{t}(\tau,\sigma),\hat{p}_t(\tau,\tilde{\sigma})] = \mathrm{i}\delta(\sigma-\tilde{\sigma}),\\
&[\hat{x}(\tau,\sigma),\hat{p}_x(\tau,\tilde{\sigma})] = \mathrm{i}\delta(\sigma-\tilde{\sigma}),\\
&[\hat{\phi}(\tau,\sigma),\hat{P}_\phi(\tau,\tilde{\sigma})] = \mathrm{i}\delta(\sigma-\tilde{\sigma}).
\end{aligned}\tag{2.39}$$

We turn now to the derivation of the explicit form for the modified first-class constraints $\tilde{\mathcal{H}}_\perp$ and $\tilde{\mathcal{H}}_1$, by observing that the formalism requires that their algebra should now close relative to the Dirac brackets. This can be achieved by further noting that

$$\{\tilde{t},\tilde{x}\}^* = 0,\tag{2.40}$$

where

$$\tilde{t} = t + \frac{\theta}{2}p_x, \qquad \tilde{x} = x - \frac{\theta}{2}p_t.\tag{2.41}$$

This selection of $\tilde{t}, \tilde{x}$ variables is not unique, since there exist an infinite number of possible choices all of which are related by canonical transformations that leave invariant the symplectic structure (2.37). At the quantum level, however, only those theories which are related by linear canonical transformations will be equivalent.

Now, taking into account that the Dirac-bracket algebra of the variables $(\tilde{t},\tilde{x},\phi,p_t,p_x,P_\phi)$ is the same as the Poisson algebra of $(t,x,\phi,p_t,p_x,P_\phi)$, it therefore follows that by setting $\tilde{\mathcal{H}}_{(\perp,1)}(z^a) = \mathcal{H}_{(\perp,1)}(\tilde{z}^a)$ we immediately have

$$\begin{aligned}
&\{\tilde{\mathcal{H}}_\perp(\tau,\sigma),\tilde{\mathcal{H}}_\perp(\tau,\sigma'\}^* = (\tilde{\mathcal{H}}_1(\tau,\sigma)+\tilde{\mathcal{H}}_1(\tau,\sigma'))\delta'(\sigma-\sigma'),\\
&\{\tilde{\mathcal{H}}_1(\tau,\sigma),\tilde{\mathcal{H}}_1(\tau,\sigma')\}^* = (\tilde{\mathcal{H}}_1(\tau,\sigma)+\tilde{\mathcal{H}}_1(\tau,\sigma'))\delta'(\sigma-\sigma'),\\
&\{\mathcal{H}_1(\tau,\sigma),\mathcal{H}_\perp(\tau,\sigma')\}^* = (\tilde{\mathcal{H}}_\perp(\tau,\sigma))\delta'(\sigma-\sigma'),
\end{aligned}\tag{2.42}$$

with

$$\begin{aligned}
\tilde{\mathcal{H}}_\perp = (-\tilde{\gamma})^{-\frac{1}{2}}\Bigg[&\frac{1}{2}(P_\phi^2+\phi'^2) + p_t\left(x-\frac{\theta}{2}p_t\right)' + p_x\left(t+\frac{\theta}{2}p_x\right)'\\
&+\left(\left(x-\frac{\theta}{2}p_t\right)'^2 - \left(t+\frac{\theta}{2}p_x\right)'^2\right)V(\phi)\Bigg] \approx 0,
\end{aligned}\tag{2.43}$$

$$\tilde{\mathcal{H}}_1 = p_x\left(x-\frac{\theta}{2}p_t\right)' + p_t\left(t+\frac{\theta}{2}p_x\right)' + P_\phi\phi' \approx 0.$$

It is interesting to note here that constraints (2.18) also appear in string theory, see, e.g., [10, 11]. One could then ask if there might also be a string theory from where constraints (2.43) would arise or if it is possible to apply the deformations proposed in our paper to the string and see if from such a procedure a NCOS theory could appear [19, 20].

When quantizing, the constraints $\tilde{\mathcal{H}}_{\perp,1}$ are promoted to the rank of operators satisfying the subsidiary conditions

$$\hat{\tilde{\mathcal{H}}}_\perp|\Psi\rangle = 0, \qquad \hat{\tilde{\mathcal{H}}}_1|\Psi\rangle = 0.\tag{2.44}$$

Also for consistency we need that at the quantum level the additional condition

$$[\hat{\tilde{\mathcal{H}}}_\perp,\hat{\tilde{\mathcal{H}}}_1]|\Psi\rangle = 0\tag{2.45}$$

be satisfied. This implies that the commutator of the first-class constraint operators has to be of the form

$$[\hat{\tilde{\mathcal{H}}}_\perp(\tau,\sigma),\hat{\tilde{\mathcal{H}}}_1(\tau,\sigma')] = \hat{c}_\perp(\sigma,\sigma')\hat{\tilde{\mathcal{H}}}_\perp + \hat{c}_1(\sigma,\sigma')\hat{\tilde{\mathcal{H}}}_1,\tag{2.46}$$

where, in general, $\hat{c}_{(\perp,1)}$ are functions of the field operators that need to appear to the left of $\hat{\tilde{\mathcal{H}}}_{(\perp,1)}$. This, in turn, involves finding the operator ordering needed to achieve this requirement in order to have an appropriate quantum theory. In the present case this does not constitute an important issue, since ordering for the super-Hamiltonian is immaterial and the difference in placing the momenta to the right or to the left of the coordinates in the super-momentum leads to a term which in the basis $|t(\sigma), p_x(\sigma), \phi(\sigma)\rangle$ (see the paragraph following equation (2.50)) is of the form $\partial_\sigma \delta(\sigma-\sigma')|_{\sigma=\sigma'}\Psi(t(\sigma), p_x(\sigma), \phi(x(\sigma), t(\sigma)), \tau)$ and which, because of the antisymmetry of the delta-function derivative, can be put equal to zero. We therefore choose the following ordering for $\hat{\tilde{\mathcal{H}}}_{(\perp,1)}$:

$$\begin{aligned}\hat{\tilde{\mathcal{H}}}_\perp = (-\tilde{\gamma})^{-\frac{1}{2}}\Bigg[&\frac{1}{2}\left(\hat{P}_\phi{}^2 + \hat{\phi}'^2\right) + \hat{p}_t\left(\hat{x} - \frac{\theta}{2}\hat{p}_t\right)' + \hat{p}_x\left(\hat{t} + \frac{\theta}{2}\hat{p}_x\right)' \\ &- \left(\left(\hat{t} + \frac{\theta}{2}\hat{p}_x\right)'^2 - \left(\hat{x} - \frac{\theta}{2}\hat{p}_t\right)'^2\right) V\left(\hat{\phi}\right)\Bigg] \approx 0, \end{aligned} \tag{2.47}$$

$$\hat{\tilde{\mathcal{H}}}_1 = \hat{p}_x\left(\hat{x} - \frac{\theta}{2}\hat{p}_t\right)' + \hat{p}_t\left(\hat{t} + \frac{\theta}{2}\hat{p}_x\right)' + \hat{P}_\phi\hat{\phi}' \approx 0.$$

Making repeated use of the identity

$$f(\sigma')\delta'(\sigma-\sigma') = f'(\sigma)\delta(\sigma-\sigma') + f(\sigma)\delta'(\sigma-\sigma') \tag{2.48}$$

in the evaluation of the commutator of these two operators, we get

$$\begin{aligned}
&2\hat{P}_\phi(\sigma)\hat{P}_\phi(\sigma')\delta'(\sigma-\sigma') = (\hat{P}_\phi^2(\sigma) + \hat{P}_\phi^2(\sigma'))\delta'(\sigma-\sigma'),\\
&2\hat{\tilde{x}}'(\sigma)\hat{\tilde{x}}'(\sigma')\delta'(\sigma-\sigma') = (\hat{\tilde{x}}'^2(\sigma) + \hat{\tilde{x}}'^2(\sigma'))\delta'(\sigma-\sigma'),\\
&2\hat{\tilde{t}}'(\sigma)\hat{\tilde{t}}'(\sigma')\delta'(\sigma-\sigma') = (\hat{\tilde{t}}'^2(\sigma) + \hat{\tilde{t}}'^2(\sigma'))\delta'(\sigma-\sigma'),\\
&(\hat{p}_t(\sigma)\hat{\tilde{x}}'(\sigma') + \hat{p}_t(\sigma')\hat{\tilde{x}}'(\sigma))\delta'(\sigma-\sigma') = (\hat{p}_t(\sigma)\hat{\tilde{x}}'(\sigma) + \hat{p}_t(\sigma')\hat{\tilde{x}}'(\sigma'))\delta'(\sigma-\sigma'),\\
&(\hat{p}_x(\sigma)\hat{\tilde{t}}'(\sigma') + \hat{p}_x(\sigma')\hat{\tilde{t}}'(\sigma))\delta'(\sigma-\sigma') = (\hat{p}_x(\sigma)\hat{\tilde{t}}'(\sigma) + \hat{p}_x(\sigma')\hat{\tilde{t}}'(\sigma'))\delta'(\sigma-\sigma'),\\
&(\hat{\tilde{x}}'^2(\sigma) + \hat{\tilde{t}}'^2(\sigma))[V(\hat{\phi}(\sigma)), \hat{P}_\phi(\sigma')]\phi'(\sigma') = \mathrm{i}(\hat{\tilde{x}}'^2(\sigma') + \hat{\tilde{t}}'^2(\sigma'))\partial_\sigma V(\hat{\phi}(\sigma))\delta(\sigma-\sigma')\\
&\qquad = \mathrm{i}(\hat{\tilde{x}}'^2(\sigma') + \hat{\tilde{t}}'^2(\sigma'))(V(\hat{\phi}(\sigma')) - V(\hat{\phi}(\sigma)))\delta'(\sigma-\sigma').
\end{aligned} \tag{2.49}$$

From these relations it follows that

$$[\hat{\tilde{\mathcal{H}}}_\perp(\tau,\sigma), \hat{\tilde{\mathcal{H}}}_1(\tau,\sigma')] = \mathrm{i}(\hat{\tilde{\mathcal{H}}}_\perp(\tau,\sigma) + \hat{\tilde{\mathcal{H}}}_\perp(\tau,\sigma'))\delta'(\sigma-\sigma'). \tag{2.50}$$

Hence our choice (2.47) is indeed of the form (2.46) and results in an appropriate Dirac quantization of the theory. In this parametrized quantization all the dynamics is hidden in the constraints although, because of the noncommutativity of the coordinate field operators $t(\tau,\sigma), x(\tau,\sigma)$, we cannot construct configuration space-state functionals of the form $\Psi[t(\sigma), x(\sigma), \phi(\sigma), \tau] = \langle t(\sigma), x(\sigma), \phi(\sigma)|\Psi(\tau)\rangle$ with the usual interpretation of a probability amplitude that the scalar field ϕ has a definite distribution $\phi(\sigma)$ on a curved spacelike hypersurface defined by $t = t(\sigma), x = x(\sigma)$ at time τ. (Note that in the Schrödinger picture the dynamical variables do not depend on τ.) We can, however, construct state amplitudes from mixed momenta and reduced configuration space eigenkets such as $|t(\sigma), p_x(\sigma), \phi(\sigma)\rangle$. In this basis $\hat{x}$ and $\hat{p}_t$ are represented by

$$\hat{x} = \mathrm{i}\left(\frac{\delta}{\delta p_x(\sigma)} - \theta\frac{\delta}{\delta t(\sigma)}\right), \tag{2.51}$$

$$\hat{p}_t = -\mathrm{i}\frac{\delta}{\delta t(\sigma)}, \tag{2.52}$$

so that from (2.47) we get

$$\left(\frac{\theta}{2}\frac{\partial}{\partial\sigma}\frac{\delta^2}{\delta t(\sigma)\delta t(\sigma)} - \frac{\partial}{\partial\sigma}\frac{\delta}{\delta t(\sigma)}\frac{\delta}{\delta p_x(\sigma)}\right)\Psi[t(\sigma), p_x(\sigma), \phi(\sigma)]$$
$$= \left[\frac{1}{2}\left(-\frac{\delta^2}{\delta\phi(\sigma)\delta\phi(\sigma)} + \phi'^2\right) + p_x\left(t' + \frac{\theta}{2}p_x'\right)\right.$$
$$\left. - \left(\left(t' + \frac{\theta}{2}p_x'\right)^2 + \frac{\partial^2}{\partial\sigma^2}\left(\frac{\delta}{\delta p_x(\sigma)} - \frac{\theta}{2}\frac{\delta}{\delta t(\sigma)}\right)^2\right)V(\phi)\right]\Psi \tag{2.53}$$

and

$$\left[p_x\frac{\partial}{\partial\sigma}\left(\frac{\delta}{\delta p_x(\sigma)} - \frac{\theta}{2}\frac{\delta}{\delta t(\sigma)}\right) - \left(t' - \frac{\theta}{2}p_x'\right)\frac{\delta}{\delta t(\sigma)} - \phi'\frac{\delta}{\delta\phi}\right]\Psi[t(\sigma), p_x(\sigma), \phi(\sigma)] = 0. \tag{2.54}$$

Thus, introducing noncommutativity by parametrizing the action in the Dirac first quantization of the scalar-field scheme leads us necessarily to the above two-fold infinity of coupled equations. Equations (2.53) and (2.54) are the analogous of the Wheeler–De Witt equations for our noncommutative scalar field, and they cannot be reduced to a Schrödinger-like equation as in the commutative case, because here we cannot solve explicitly the super-Hamiltonian and super-momentum constraints for the momenta p_t and p_x. It is not our objective here to investigate this system any further. It is important however to note that the non-locality of the theory is reflected in these equations, so extreme care is required when regularizing these expressions in such a way that this regularization does not break the deformed symmetries generated by the constraints. Furthermore, the definition of the scalar product of the wave functionals $\Psi[t(\sigma), p_x(\sigma), \phi(\sigma)]$ must imply the unitarity of the theory.

In the following, we analyze the deformed symmetries which result from the deformed constraints of the theory, which in turn result from the spacetime noncommutativity, and derive a general anzatz for constructing these deformed symmetries for any field theory.

3. Spacetime noncommutativity and deformed symmetries

We have seen that the Dirac-bracket algebra (2.42) together with (2.43) provides an algorithm for constructing the deformed gauge symmetries associated with the reparametrization invariance of action (2.3), where a symplectic structure was introduced in order to allow for the appearance of spacetime noncommutativity when applying Dirac's procedure for canonical quantization to the original action. In fact, making use of (2.36) one can show that

$$\{t^n(\tau,\sigma), x^m(\tau,\sigma')\}^* = nm\theta t^{n-1}(\tau,\sigma)x^{m-1}(\tau,\sigma')\delta(\sigma-\sigma'). \tag{3.1}$$

On the other hand, evaluating the Moyal product $(x^\mu)^n \star_\theta (x^\nu)^m$ with the bidifferential

$$\star_\theta := \exp\left[\frac{\mathrm{i}}{2}\theta^{\mu\nu}\int \mathrm{d}\sigma''\frac{\overleftarrow{\delta}}{\delta x^\mu(\tau,\sigma'')}\frac{\overrightarrow{\delta}}{\delta x^\nu(\tau,\sigma'')}\right], \tag{3.2}$$

and comparing with (3.1), we have that

$$\{t^n(\tau,\sigma), x^m(\tau,\sigma')\}^* \cong [t^n(\tau,\sigma), x^m(\tau,\sigma')]_{\star_\theta}$$
$$:= t^n(\tau,\sigma)\star_\theta x^m(\tau,\sigma') - x^m(\tau,\sigma')\star_\theta t^n(\tau,\sigma). \tag{3.3}$$

More generally, for Dirac brackets of arbitrary $A(\tau,\sigma), B(\tau,\sigma)$ functionals of $t(\tau,\sigma), p_t(\tau,\sigma), x(\tau,\sigma), p_x(\tau,\sigma), \phi(\tau,\sigma)$ and $P_\phi(\tau,\sigma)$ we get

$$\{A(\tau,\sigma), B(\tau,\sigma')\}^* \cong [A(\tau,\sigma), B(\tau,\sigma')]_{\star_\theta}, \tag{3.4}$$

after identifying the momenta in the left-hand side of the above equation with their corresponding differential operators on the right-hand side. We thus have a morphism from the Poisson–Dirac algebra of functionals of t, x, ϕ, p_t, p_x and P_ϕ, to the algebra of differential operators obtained from these functionals (after making the maps $p_t \mapsto -\mathrm{i}\delta/\delta_t$, $p_x \mapsto -\mathrm{i}\delta/\delta_x$, $P_\phi \mapsto -\mathrm{i}\delta/\delta_\phi$) with multiplication given by the $\star_\theta$-product commutator.

Applying now the above-described algebra morphism to (2.42) results in

$$\begin{aligned}
[\tilde{\mathcal{H}}^\star_\perp(\tau,\sigma), \tilde{\mathcal{H}}^\star_\perp(\tau,\sigma')]_{\star\theta} &= (\tilde{\mathcal{H}}^\star_1(\tau,\sigma) + \tilde{\mathcal{H}}^\star_1(\tau,\sigma'))\delta'(\sigma-\sigma'),\\
[\tilde{\mathcal{H}}^\star_1(\tau,\sigma), \tilde{\mathcal{H}}^\star_1(\tau,\sigma')]_{\star\theta} &= (\tilde{\mathcal{H}}^\star_1(\tau,\sigma) + \tilde{\mathcal{H}}^\star_1(\tau,\sigma'))\delta'(\sigma-\sigma'),\\
[\tilde{\mathcal{H}}^\star_1(\tau,\sigma), \tilde{\mathcal{H}}^\star_\perp(\tau,\sigma')]_{\star\theta} &= (\tilde{\mathcal{H}}^\star_\perp(\tau,\sigma))\delta'(\sigma-\sigma').
\end{aligned} \tag{3.5}$$

Here, the notation $\tilde{\mathcal{H}}^\star_{(\perp,1)}$ stands for the differential operators

$$\tilde{\mathcal{H}}^\star_{\perp,1}(\tau,\sigma) := \mathcal{H}_{\perp,1}(\tau,\sigma)\exp\left[-\frac{\mathrm{i}}{2}\theta^{\mu\nu}\int \mathrm{d}\sigma'' \frac{\overleftarrow{\delta}}{\delta x^\mu(\tau,\sigma'')}\frac{\overrightarrow{\delta}}{\delta x^\nu(\tau,\sigma'')}\right], \tag{3.6}$$

and their algebra multiplication μ_θ is given by

$$\mu_\theta(\tilde{\mathcal{H}}^\star_i \otimes \tilde{\mathcal{H}}^\star_j) = \tilde{\mathcal{H}}^\star_i \star \tilde{\mathcal{H}}^\star_j, \qquad i, j = \perp, 1. \tag{3.7}$$

Note that from (3.6) it follows that

$$[\tilde{\mathcal{H}}^\star_i(\tau,\sigma), \tilde{\mathcal{H}}^\star_j(\tau,\sigma')]_{\star\theta} = [\mathcal{H}_i(\tau,\sigma), \mathcal{H}_j(\tau,\sigma')]\,\mathrm{e}^{[-\frac{\mathrm{i}}{2}\theta^{\mu\nu}\int \mathrm{d}\sigma'' \frac{\overleftarrow{\delta}}{\delta x^\mu(\tau,\sigma'')}\frac{\overrightarrow{\delta}}{\delta x^\nu(\tau,\sigma'')}]}, \qquad i, j = \perp, 1 \tag{3.8}$$

and substituting (3.6) and (3.8) into (3.5) we get

$$\begin{aligned}
[\mathcal{H}_\perp(\tau,\sigma), \mathcal{H}_\perp(\tau,\sigma')] &= (\mathcal{H}_1(\tau,\sigma) + \mathcal{H}_1(\tau,\sigma'))\delta'(\sigma-\sigma'),\\
[\mathcal{H}_1(\tau,\sigma), \mathcal{H}_1(\tau,\sigma')] &= (\mathcal{H}_1(\tau,\sigma) + \mathcal{H}_1(\tau,\sigma'))\delta'(\sigma-\sigma'),\\
[\mathcal{H}_1(\tau,\sigma), \mathcal{H}_\perp(\tau,\sigma')] &= (\mathcal{H}_\perp(\tau,\sigma))\delta'(\sigma-\sigma'),
\end{aligned} \tag{3.9}$$

which is the algebra of differential operator generators isomorphic to the non-deformed algebra (2.21).

Furthermore, since by (2.18)

$$\begin{aligned}
\{\phi, \mathcal{H}_\perp\} &= (-\gamma)^{-\frac{1}{2}}\left[\frac{(x'^2 - t'^2)}{\sqrt{-g}}\dot{\phi} + \frac{(t'\dot{t} - x'\dot{x})}{\sqrt{-g}}\phi'\right],\\
\{\phi, \mathcal{H}_1\} &= \phi',
\end{aligned} \tag{3.10}$$

the generators $\mathcal{H}_i$ of (2.21)—the Virasoro algebra $\mathcal{V}$—can be viewed as derivations acting on elements $\phi(t(\tau,\sigma), x(\tau,\sigma))$ of the algebra of functions $\mathcal{A}$, with point multiplication μ. That is,

$$\begin{aligned}
\{\phi, \mathcal{H}_\perp\} &\cong \hat{\mathcal{H}}_\perp \triangleright \phi = (-\gamma)^{-\frac{1}{2}}\left(\frac{(x'^2 - t'^2)}{\sqrt{-g}}\partial_\tau + \frac{(t'\dot{t} - x'\dot{x})}{\sqrt{-g}}\partial_\sigma\right) \triangleright \phi\\
\{\phi, \hat{\mathcal{H}}_1\} &\cong \hat{\mathcal{H}}_1 \triangleright \phi = \partial_\sigma \triangleright \phi.
\end{aligned} \tag{3.11}$$

In addition, since $\hat{\mathcal{H}}_i \in \hat{\mathcal{V}}$ is a (infinite-dimensional) Lie algebra, its universal envelope $U(\hat{\mathcal{V}})$ can be given by the structure of a Hopf algebra with coproduct

$$\Delta(\hat{\mathcal{H}}_i) = \hat{\mathcal{H}}_i \otimes 1 + 1 \otimes \hat{\mathcal{H}}_i, \qquad i = \perp, 1 \tag{3.12}$$

and antipode

$$S(\hat{\mathcal{H}}_i) = -\hat{\mathcal{H}}_i, \qquad i = \perp, 1, \tag{3.13}$$

so $\mathcal{A}$ is a left module algebra over $U(\hat{\mathcal{V}})$. In parallel, for the symplectic structure (2.37) we have the algebra $\hat{\mathcal{V}}^{\star}$ of derivation operators $\hat{\tilde{\mathcal{H}}}_i^{\star}$, defined in analogy to (3.6) by

$$\hat{\tilde{\mathcal{H}}}^{\star}_{\perp,1}(\tau,\sigma) := \hat{\mathcal{H}}_{\perp,1}(\tau,\sigma)\exp\left[-\frac{\mathrm{i}}{2}\theta^{\mu\nu}\int \mathrm{d}\sigma''\frac{\overleftarrow{\delta}}{\delta x^{\mu}(\tau,\sigma'')}\frac{\overrightarrow{\delta}}{\delta x^{\nu}(\tau,\sigma'')}\right], \tag{3.14}$$

with multiplication μ_θ generated by (3.4), and the corresponding left module algebra $\mathcal{A}_\theta$ over $U(\hat{\mathcal{V}}^{\star})$, whose elements are now functions $\phi(t(\tau,\sigma),x(\tau,\sigma))$ with multiplication μ_θ inherited from (3.3).

From (3.14) it immediately follows that

$$\hat{\tilde{\mathcal{H}}}^{\star}_i \star_\theta \phi(t,x) = \hat{\mathcal{H}}_i \triangleright \phi(t,x), \tag{3.15}$$

so the action of elements of the twisted algebra $\hat{\mathcal{V}}^{\star}$ on elements of $\mathcal{A}_\theta$ is equal to the action of the corresponding elements of the untwisted algebra on the corresponding elements of the ordinary algebra $\mathcal{A}$ of functions of commuting variables. Thus, the morphism from $\hat{\mathcal{V}}$ to $\hat{\mathcal{V}}^{\star}$ by

$$\hat{\mathcal{H}}_i \mapsto \hat{\tilde{\mathcal{H}}}^{\star}_i \tag{3.16}$$

induces the morphism from $\mathcal{A}$ to $\mathcal{A}_\theta$ by

$$\mu(f(t,x)\otimes g(t,x)) \mapsto \mu_\theta(f(t,x)\otimes g(t,x)). \tag{3.17}$$

It should now be fairly obvious, by mere observation of the notation already introduced, how our algorithm can be readily generalized to higher dimensional noncommutative spacetimes with constant parameters of noncommutativity. Thus, the commutator relations for the spacetime coordinate fields at equal times will now be given by

$$[x^{\mu}(\tau,\boldsymbol{\sigma}), x^{\nu}(\tau,\boldsymbol{\sigma}')] = \mathrm{i}\theta^{\mu\nu}\delta^{D}(\boldsymbol{\sigma}-\boldsymbol{\sigma}'), \tag{3.18}$$

where $\theta^{\mu\nu}$ = const. As in the bi-dimensional spacetime case, in the extended $(2D+4)$-dimensional reparametrized phase space with a general symplectic structure we can also derive the algebra of constraints by introducing a new set of commuting coordinate fields defined by

$$\tilde{x}^{\mu}(\boldsymbol{\sigma}) = x^{\mu}(\boldsymbol{\sigma}) + \frac{\theta^{\mu\nu}}{2}p_\nu(\boldsymbol{\sigma}). \tag{3.19}$$

The new constraints will then have the form

$$\begin{aligned}\tilde{\mathcal{H}}_\perp &= \tfrac{1}{2}\left(P_\phi^2 + \tilde{\gamma}\tilde{\gamma}^{ij}\partial_{\sigma^i}\phi\partial_{\sigma^j}\phi\right) + \sqrt{-\tilde{\gamma}}\tilde{n}^{\nu}p_\nu - \tilde{\gamma}V(\phi) \approx 0,\\ \tilde{\mathcal{H}}_i &= P_\phi\partial_{\sigma^i}\phi + p_\mu\partial_{\sigma^i}\tilde{x}^{\mu}.\end{aligned} \tag{3.20}$$

Furthermore, making use of the algebra morphism discussed at the beginning of this section we then arrive at the twisted algebra

$$\begin{aligned}&[\tilde{\mathcal{H}}^{\star}_\perp(\tau,\boldsymbol{\sigma}), \tilde{\mathcal{H}}^{\star}_\perp(\tau,\boldsymbol{\sigma}')]_{\star\theta} = \sum_{i=1}^{D}(\tilde{\mathcal{H}}^{\star}_i(\tau,\boldsymbol{\sigma}) + \tilde{\mathcal{H}}^{\star}_i(\tau,\boldsymbol{\sigma}'))\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),\\ &[\tilde{\mathcal{H}}^{\star}_i(\tau,\boldsymbol{\sigma}), \tilde{\mathcal{H}}^{\star}_j(\tau,\boldsymbol{\sigma}')]_{\star\theta} = (\tilde{\mathcal{H}}^{\star}_i(\tau,\boldsymbol{\sigma})\partial_{\sigma^j}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}') + \tilde{\mathcal{H}}^{\star}_j(\tau,\boldsymbol{\sigma}')\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}')),\\ &[\tilde{\mathcal{H}}^{\star}_i(\tau,\boldsymbol{\sigma}), \tilde{\mathcal{H}}^{\star}_\perp(\tau,\boldsymbol{\sigma}')]_{\star\theta} = (\tilde{\mathcal{H}}^{\star}_\perp(\tau,\boldsymbol{\sigma}))\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}').\end{aligned} \tag{3.21}$$

Let us now study the deformed symmetries generated by algebra (3.21). For this purpose and also in order to make contact with some related results appearing in the literature, consider the canonical transformation

$$\begin{aligned}
H_\tau[\xi] &= \int \mathrm{d}\vec{\sigma}\xi^\alpha(\tau,\vec{\sigma})H_\alpha(\tau,\vec{\sigma}) \\
&= \int \mathrm{d}\vec{\sigma}(\xi^\perp(\tau,\vec{\sigma})n^\alpha + \xi^a(\tau,\vec{\sigma})s_a^\alpha)H_\alpha \\
&= \int \mathrm{d}\vec{\sigma}(\xi^\perp(\tau,\sigma)\mathcal{H}_\perp(\tau,\sigma) + \xi^a(\tau,\sigma)\mathcal{H}_a(\tau,\sigma)), \qquad a = 1,\ldots,D.
\end{aligned} \tag{3.22}$$

Thus, for the extended parametrized fields we have

$$\begin{aligned}
\delta x^\mu &= \{x^\mu(\tau,\vec{\sigma}), H_\tau[\xi]\} \cong \hat{H}_\tau[\xi] \triangleright x^\mu \\
&= \int \mathrm{d}\vec{\sigma}'\{x^\mu(\tau,\vec{\sigma}), \xi^\perp(\tau,\vec{\sigma}')n^\alpha(\tau,\vec{\sigma}')p_\alpha(\tau,\vec{\sigma}') + \xi^a(\tau,\vec{\sigma}')s_a^\alpha(\tau,\vec{\sigma}')p_\alpha(\tau,\vec{\sigma}')\} \\
&= (\xi^\perp n^\alpha + \xi^a s_a^\alpha)\delta_\alpha^\mu = \xi^\alpha(\tau,\vec{\sigma})\frac{\partial x^\mu}{\partial x^\alpha},
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
\delta\phi(x(\tau,\vec{\sigma})) &= \{\phi(x(\tau,\vec{\sigma})), H_\tau[\xi]\} \cong \hat{H}_\tau[\xi] \triangleright \phi(x(\tau,\vec{\sigma})) \\
&= \frac{\xi^\perp\sqrt{-g}}{\sqrt{-\gamma}}g^{0\alpha}\frac{\partial\phi}{\partial\sigma^\alpha} + \xi^a\frac{\partial\phi}{\partial\sigma^a} = (\text{by (2.8)}) \\
&= \left(\xi^\perp n^\beta + \xi^a s_a^\beta\right)\frac{\partial\phi}{\partial x^\beta} = \xi^\beta(\tau,\vec{\sigma})\frac{\partial\phi(x(\tau,\vec{\sigma}))}{\partial x^\beta}.
\end{aligned} \tag{3.24}$$

Consequently the derivations $\hat{H}_\tau[\xi] \equiv \delta_\xi = \xi^\alpha(x(\tau,\vec{\sigma}))\frac{\partial}{\partial x^\alpha}|_{x(\tau,\vec{\sigma})}$ can be seen as the complete vector fields in the embedding $x(\tau,\vec{\sigma})$ of the generators

$$\xi(x) := \xi^\alpha(x)\frac{\partial}{\partial x^\alpha} \in \mathrm{Ldiff}\,\mathcal{M}.$$

of the Lie algebra of spacetime diffeomorphisms.

Moreover, since $[\eta,\rho] = \pounds_\eta\rho$ we have

$$\begin{aligned}
[\delta_\eta,\delta_\rho]\phi &= \delta_{\pounds_\eta\rho}\phi = \hat{H}_\tau[\pounds_\eta\rho] \triangleright \phi \cong \{\phi, H_\tau[\pounds_\eta\rho]\} \\
&= \hat{H}_\tau[\eta] \triangleright [\hat{H}_\tau[\rho] \triangleright \phi] - \hat{H}_\tau[\rho] \triangleright [\hat{H}_\tau[\eta] \triangleright \phi] \\
&\cong \{\{\phi, H_\tau[\rho]\}, H_\tau[\eta]\} - \{\{\phi, H_\tau[\eta]\}, H_\tau[\rho]\} = -\{\phi, \{H_\tau[\eta], H_\tau[\rho]\}\},
\end{aligned} \tag{3.25}$$

after making use of the Jacobi identity. We therefore have the known (see, e.g., [18]) anti-homomorphism between the Poisson algebra $\mathcal{V}$, generated by (3.22), and the Lie algebra of spacetime diffeomorphisms.

In going over to the noncommutative spacetime case, we proceed according to our previously derived algorithm, i.e. we replace the Poisson brackets by Dirac brackets and $t \to \tilde{t}$, $x \to \tilde{x}$. Hence, we can now write the functorial diagrams

$$\begin{array}{ccc}
H_\tau[\xi] \in \mathcal{V} & \xrightarrow{\theta} & \mathcal{V}^\star \ni \tilde{H}_\tau^\star[\tilde{\xi}] = \int d\boldsymbol{\sigma}(\tilde{\xi}^\perp\tilde{\mathcal{H}}_\perp^\star + \tilde{\xi}^a\tilde{\mathcal{H}}_a^\star) \\
\mathcal{C}\Big\downarrow & & \mathcal{C}\Big\downarrow \\
\hat{H}_\tau[\xi] \in \hat{\mathcal{V}} & \xrightarrow{\mathcal{C}(\theta)} & \hat{\mathcal{V}}^\star \ni \hat{\tilde{H}}_\tau^\star[\tilde{\xi}] \equiv \delta_\xi^\star;
\end{array} \tag{3.26}$$

$$\begin{array}{ccc}
\phi \in \mathcal{A} & \xrightarrow{\delta_\xi} & \mathcal{A} \ni \quad \delta_\xi \triangleright \phi \\
\mathcal{D}\Big\downarrow & & \mathcal{D}\Big\downarrow \\
\phi \in \mathcal{A}_\theta & \xrightarrow{\mathcal{D}(\delta_\xi^\star)} & \mathcal{A}_\theta \ni \delta_\xi^\star \triangleright \phi = \delta_\xi^\star \star \phi(x(\tau,\boldsymbol{\sigma}));
\end{array} \tag{3.27}$$

i.e.

$$\{\phi, H_\tau[\xi]\} \cong \delta_\xi \triangleright \phi \mapsto \delta^\star_\xi \star \phi(x(\tau, \boldsymbol{\sigma})). \tag{3.28}$$

Note that equations (3.26)–(3.28) provide an explicit expression for the mapping $\delta_\rho \mapsto \delta^\star_\rho$, which in turn imply

$$[\delta^\star_\rho, \delta^\star_\eta]_{\star\theta} = \delta^\star_{\pounds_\rho \eta}, \tag{3.29}$$

$$\delta^\star_\rho \star (f \star g) = \delta_\rho (f \star g). \tag{3.30}$$

We can now compare some of our results with those obtained in [21]. Thus, we have that our equation (3.14) for the twisted derivations $\hat{\tilde{\mathcal{H}}}^\star_\alpha$ corresponds to equation (3.26) in [21], while algebra (3.29) and the derivation $\delta^\star_\rho$ correspond to equations (5.3) and (5.4) there. Note also that since we had previously shown that the universal envelope $U(\hat{\mathcal{V}})$ had the structure of a Hopf algebra, the above morphisms imply that $U(\hat{\mathcal{V}}^\star)$ is also a Hopf algebra. We can obtain an explicit expression for the coproduct by making use of the duality between the product and coproduct, followed by the application of equations (3.30) and (3.6). Thus we get

$$\begin{aligned}
\mu_\theta \circ \Delta(\delta^\star_\rho)(f \otimes g) &= \delta^\star_\rho \star (f \star g) = \delta_\rho (f \star g) = \mu(\delta_\rho \otimes 1 + 1 \otimes \delta_\rho)\big(\mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g\big) \\
&= \sum_n \frac{1}{n!}\left(\frac{\mathrm{i}}{2}\right)^n \theta^{\mu_1\nu_1} \cdots \theta^{\mu_n\nu_n}\Big[\big(\delta^\star_\rho \star \partial_{\mu_1\cdots\mu_n} f\big)\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} \star \partial_{\nu_1\cdots\nu_n} g \\
&\quad + \big(\partial_{\mu_1\cdots\mu_n} f\big)\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} \star \big(\delta^\star_\rho \star \partial_{\nu_1\cdots\nu_n} g\big)\Big] \\
&= \mu_\theta \circ \Big[\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}(\delta^\star_\rho \otimes 1 + 1 \otimes \delta^\star_\rho)\, \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}\Big](f \otimes g).
\end{aligned} \tag{3.31}$$

This result also compares with the Leibniz rule given by equation (5.9) in [21]. Further note that if we let $\mathcal{F} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \in U(\hat{\mathcal{V}}) \otimes U(\hat{\mathcal{V}})$ and define $f \star g = \mu_\theta(f \otimes g) := \mu(\mathcal{F}^{-1} \triangleright (f \otimes g))$, we then have

$$\begin{aligned}
\delta_\rho(f \star g) &= \delta_\rho \triangleright \mu(\mathcal{F}^{-1} \triangleright (f \otimes g)) = \mu[(\Delta\delta_\rho)\mathcal{F}^{-1} \triangleright (f \otimes g)] \\
&= \mu\mathcal{F}^{-1}[(\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})(f \otimes g)] \\
&= \mu_\theta[(\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})(f \otimes g)].
\end{aligned} \tag{3.32}$$

Thus, the undeformed coproduct of the symmetry Hopf algebra $U(\hat{\mathcal{V}})$ is related to the Drinfeld twist $\Delta^{\mathcal{F}}$ by the inner endomorphism $\Delta^{\mathcal{F}}\delta_\rho := (\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})$ and, by (3.32), it preserves the covariance:

$$\begin{aligned}
\delta_\rho \triangleright (f \cdot g) &= \mu \circ [\Delta(\delta_\rho)(f \otimes g)] = (\delta_{\rho(1)} \triangleright f) \cdot (\delta_{\rho(2)} \triangleright g) \\
&\xrightarrow{\theta} \delta^\star_\rho \triangleright (f \star g) = (\delta^\star_{\rho(1)} \triangleright f) \star (\delta^\star_{\rho(2)} \triangleright g),
\end{aligned} \tag{3.33}$$

where we have used the Sweedler notation for the coproduct. Consequently, the twisting of the coproduct is tied to the deformation $\mu \to \mu_\theta$ of the product when the last one is defined by

$$f \star g := \big(\mathcal{F}^{-1}_{(1)} \triangleright f\big)\big(\mathcal{F}^{-1}_{(2)} \triangleright g\big). \tag{3.34}$$

A more extensive discussion of the application of some of these algebras to the construction of a deformed differential geometry for gravity theories may also be found in [21] as well as other works cited therein.

If we now assume that the coefficients of the vector fields δ_ξ are linear in the spacetime variables, then the generators δ_ρ in (3.31) become the infinitesimal generators of the Poincaré transformations, and the coproduct defined in this equation reduces to the twisted coproduct considered by, e.g., [22]. We would like to stress, however, that while all the above-mentioned papers, as well as a large number of others appearing in the literature, start from equating spacetime noncommutativity with the noncommutativity of the parameters of the functions denoting classical fields, and deforming the algebra of these fields via the Moyal $\star$-product, none of the algebras $\hat{\mathcal{V}}^{\star}$ and $\mathcal{A}_\theta$ in our approach are assumed *a priori*. In contrast, they appear naturally, as does the spacetime noncommutativity, as a consequence of implementing Dirac's canonical quantization formalism for constrained systems with an arbitrary symplectic structure. Note, in particular, that in our formalism the spacetime variables are dynamical, as would be expected when viewing quantum mechanics as a minisuperspace of field theory, and their noncommutativity results from the quantization of their Dirac brackets. The deformation of the module algebra $\mathcal{A}$—in which the fields originally lived—to $\mathcal{A}_\theta$, so that by (3.16) and (3.17) functions of the field multiply according to μ_θ is in our formalism, again a consequence of the spacetime noncommutativity resulting from the quantization of the Dirac brackets and the concomitant deformation of the constraints associated with the symmetries of the field Lagrangian.

With constraints (3.20) it is possible to construct a quantum theory in the Schrödinger representation analogous to (2.53) and (2.54). As in that case, however, since these constraints are no longer linear and algebraic in the momenta (they contain mixed products of p_μ's and their derivatives), it is not possible to solve explicitly for the spacial momenta in order to construct a Schrödinger-type equation. Nonetheless, it is still possible to show that the action in the reduced configuration space is in agreement with the usually proposed noncommutative field theory for a scalar field.

4. Concluding remarks

We have shown in this paper how, by considering a reparametrized Hamiltonian canonical formulation of field theory consisting of embedding a spacial manifold Σ in the spacetime manifold, it is possible to give spacetime a dynamical character and introduce spacetime noncommutativity from first principles. We have accomplished this by resorting to an extended phase space, leading to second-class constraints which, when removed according to the Dirac quantization procedure, lead in turn to Dirac brackets. The latter then result in the deformed symplectic structure for the spacetime coordinates and corresponding canonical momenta, which yield the desired noncommutativity.

An important characteristic of our formulation is the automatic deformation of the symmetry generators when the symplectic structure is deformed. Such a deformation being imposed by the consistency conditions on the constraints (see the discussion in subsection 2.2), which have as a result that the algebra of the deformed constraints is maintained in the noncommutative case. This provides us then with a straightforward algorithm for constructing the Drinfeld twist of the Hopf algebras that one can associate with the reparametrization symmetry groups. In addition, our formalism can be readily extended to spacetimes of any dimensions and to the consideration of different possible types of deformed products, of which the Moyal product is just a particular case. Thus the formalism described here may also turn out to be useful for achieving a better understanding of noncommutative theories in curved backgrounds and twisted symmetries in Yang–Mills field theories, since in that latter case, in addition to the constraints associated with the reparametrization, we will also have the constraints associated with invariance under the gauge transformations

$$A_\mu(x) \to U(x)A_\mu(x)U^{-1}(x) + \mathrm{i}U(x)\partial_\mu U^{-1}(x),$$

so the full set must then be analyzed in order to see how it is to be twisted when noncommutativity is introduced.

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